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Abstract

A metric invariant a_k is defined, and we have that $a_k(X) \leq a_k(S^n)$ holds in an Alexandrov space X with curvature ≥ 1([So]). And the borderline case when $a_{2p-1}(X) = a_{2p-1}(S^n)$ and $a_k(S^n)$ are studied.

KEYWORDS: Metric Invariants, Alexandrov Spaces, Spheres, Borderline Cases

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SOME METRIC INVARIANTS OF SPHERES AND ALEXANDROV SPACES II

NOBUYUKI SOCHI

ABSTRACT. A metric invariant a_k is defined, and we have that $a_k(X) \leq a_k(S^n)$ holds in an Alexandrov space X with curvature ≥ 1([So]). And the borderline case when $a_{2p-1}(X) = a_{2p-1}(S^n)$ and $a_k(S^n)$ are studied.

1. INTRODUCTION

In the previous paper([So]) we introduced a metric invariant $a_k(X)$ for a compact metric space X, and studied the explicit value of $a_k(S^n)$ for an n-dimensional round sphere S^n with radius 1. Furthermore, we studied the behavior of $a_k(X)$ for an Alexandrov space X with curvature ≥ 1. In the present paper we continue to study the above invariant a_k and give answers to some problems that are conjectured in [So]. We begin with recalling the definition of a_k and results obtained in [So]. The distance between $x, y \in X$ will be denoted by $dist(x, y)$.

Definition 1.1. *For a positive integer k, we define the metric invariant $a_k(X)$ of X as follows:*

$$a_k(X) = \min_{x_1,\ldots,x_k \in X} \max_{x \in X} \frac{1}{k} \sum_{i=1}^{k} dist(x, x_i). \tag{1.1}$$

In the previous paper we were concerned with $a_k(S^n)$ and got the explicit value of $a_k(S^1)$. A k-tuple $(x_1, \cdots, x_k)$ of points $x_i (i = 1, \cdots, k)$ of S^1 located in counterclockwise order is called a configuration, where each x_i is called a vertex of the configuration.

Theorem 1.1. *(1) For $k = 2p - 1$, we have*

$$a_k(S^1) = \frac{2p^2 - 2p + 1}{(2p-1)^2}\pi. \tag{1.2}$$

$a_k(S^1)$ is realized if and only if a configuration $(x_1, \cdots, x_k)$ of k points is equally spaced in S^1, and $\max_{x \in S^1}(1/k)\sum_{i=1}^{k} dist(x, x_i)$ is attained exactly at the antipodal points of $x_i (1 \leq i \leq k)$.
(2) For $k = 2p$, we have

$$a_k(S^1) = \frac{1}{2}\pi. \tag{1.3}$$

$a_k(S^1)$ is realized if and only if $(x_1, \cdots, x_{2p})$ consists of pairs of antipodal points, and in the case we have $(1/k)\sum_{i=1}^{k} dist(x, x_i) \equiv \pi/2$.

Now in this paper we complete the following theorem for general dimension n.

Theorem 1.2. *(1) For $k = 2p - 1$, we have*

$$a_k(S^n) = a_k(S^1) = \frac{2p^2 - 2p + 1}{(2p-1)^2}\pi. \tag{1.4}$$

$a_{2p-1}(S^n)$ is realized if and only if $\{x_1, \cdots, x_{2p-1}\}$ is located on a great circle S^1 and gives an equally spaced configuration after rearranging the order of $\{x_1, \cdots, x_{2p-1}\}$, and $\max_{x \in S^n}(1/k)\sum_{i=1}^{k} dist(x, x_i)$ is attained exactly at the antipodal points of $x_i (1 \le i \le k)$.
(2) For $k = 2p$, we have

$$a_k(S^n) = \frac{1}{2}\pi. \tag{1.5}$$

Moreover, $a_k(S^n)$ is realized if and only if the set $\{x_1, \cdots, x_{2p}\}$ consists of pairs of antipodal points, and in the case we have $(1/k)\sum_{i=1}^{k} dist(x, x_i) \equiv \pi/2$.

Theorem 1.2(2) is already proved in section 4 of [So]. Theorem 1.2(1) for $k = 3$ is also proved in section 3 of [So]. We give a proof of Theorem 1.2(1) for all $k = 2p - 1$ in section 2 of this paper. In our proof another metric invariant k-extent $xt_k(X)$ introduced by Grove and Markvorsen plays an important role. Recall that for an integer $k \ge 2$, $xt_k(X)$ is defined as follows:

$$xt_k(X) = \max_{x_1, \cdots, x_k \in X} \binom{k}{2}^{-1} \sum_{i<j} dist(x_i, x_j). \tag{1.6}$$

Now we give some results of [G-M] on $xt_k(S^n)$ which are necessary for our proof.

Theorem 1.3 (Grove-Markvorsen)**.** *For all $n \ge 1$ and $k \ge 2$ we have*

$$xt_k(S^n) = xt_k(S^1) = \pi \Big/ \left(2 - \left[\frac{k+1}{2}\right]^{-1}\right). \tag{1.7}$$

Those points that realize $xt_k(S^n)$ all lie on a great circle except for antipodal pairs.

More specifically, $xt_k(S^1)$ is given as follows:

$$xt_{2p}(S^1) = \frac{p}{2p-1}\pi, \tag{1.8}$$

$$xt_{2p-1}(S^1) = \frac{p}{2p-1}\pi. \tag{1.9}$$

$xt_{2p}(S^1)$ is realized if and only if $\{x_1, \cdots, x_{2p}\}$ consists of pairs of antipodal points.

Now we explain the separation condition as follows. A configuration $(x_1, \cdots, x_k)$ in S^1 which dose not contain any pair of antipodal points is said to satisfy the separation condition if the following is satisfied: For any x_i the line through x_i and the origin separates $(x_1, \cdots, x_k) \setminus \{x_i\}$ into sets of equal cardinality. Note that k is odd in this case. Then $xt_{2p-1}(S^1)$ is realized if and only if $\{x_1, \cdots, x_{2p-1}\}$ consists of pairs of antipodal points together with a configuration of points satisfying the separation condition.

For $k = 2p-1$, $a_k(S^n)$ is related to $xt_k(S^n)$ by an inequality

$$a_k(S^n) \geq \pi - 2k^{-2}\binom{k}{2} xt_k(S^n) = \frac{2p^2-2p+1}{(2p-1)^2}\pi, \tag{1.10}$$

as is shown in section 2.

Our proof of Theorem 1.2 depends also on a theorem of K. Kiyohara in [K] : For an equally spaced configuration $(x_1, \cdots, x_n)$ of a great circle S^1 of S^2, a function $f_{x_1,\cdots,x_k}(x) = \sum_{i=1}^k dist(x, x_i)$ on S^2 attains its maximum when and only when $x = \bar{x}_l (l = 1, \cdots, k)$.

Let X be an Alexandrov space with curvature ≥ 1 and we want to compare $a_k(X)$ with $a_k(S^n)$. In the previous paper we got the following theorem by using the generalized Toponogov comparison theorem([Pe]).

Theorem 1.4. *Let X be an n-dimensional Alexandrov space with curvature ≥ 1, then we have*

$$a_k(X) \leq a_k(S^n). \tag{1.11}$$

Especially, we have

$$a_{2p}(X) \leq a_{2p}(S^n) = \frac{\pi}{2}. \tag{1.12}$$

In the case where equality holds in (1.11) for $k = 3$ we showed that X is isometric to a double spherical suspension, where the generalized Toponogov comparison theorem played an important role(see section 5 of [So]). In the present paper we improve Theorem 1.4 and show the following theorem.

Theorem 1.5. *Let X be an n-dimensional Alexandrov space with curvature ≥ 1. Suppose $a_{2p-1}(X) = a_{2p-1}(S^n)$. Then X is isometric to the unit sphere S^n.*

2. Proof of Theorem1.2(1)

In this section we give a proof of Theorem 1.2(1). First we will show for $k = 2p-1$

$$a_k(S^n) \geq \frac{2p^2-2p+1}{(2p-1)^2}\pi. \tag{2.1}$$

In the proof we apply a result of Grove-Markvorsen on k-extent $xt_k(S^n)$ of the sphere(see Theorem 1.3). Recall that for $k = 2p-1$ the k-extent $xt_k(S^n) = 1/\binom{k}{2}\max_{y_1,\cdots,y_k \in S^n}\sum_{i<j} dist(y_i, y_j)$ of the sphere is equal to $p(p-1)\pi/\binom{2p-1}{2} = p\pi/(2p-1)$. Let $x_1, \cdots, x_k$ be points on S^n that realize $a_k(S^n)$, i.e., $a_k(S^n) = \max_{x\in S^n} 1/k\sum_{i=1}^k dist(x, x_i)$. We denote by $\bar{x}$ the antipodal point of $x \in S^n$. Then for any $\bar{x}_j (1 \leq j \leq k)$ we have

$$ka_k(S^n) \geq \sum_{i=1}^k dist(x_i, \bar{x}_j). \tag{2.2}$$

Adding (2.2) with respect to j we obtain

$$\begin{aligned} k^2 a_k(S^n) &\geq \sum_{j=1}^k\sum_{i=1}^k dist(x_i, \bar{x}_j) \\ &= k^2\pi - 2\sum_{i<j} dist(x_i, x_j) \\ &\geq k^2\pi - 2\max_{y_1,\cdots,y_k\in S^n}\sum_{i<j} dist(y_i, y_j) \\ &= k^2\pi - 2\binom{k}{2}xt_k(S^n) = (2p^2-2p+1)\pi. \end{aligned} \tag{2.3}$$

Hence we have (2.1). To complete the proof of Theorem 1.2(1) we have to show the following assertions:

(1) $a_k(S^n) = (2p^2-2p+1)\pi/(2p-1)^2$.
(2) If $\{x_1, \cdots, x_k\}$ realizes $a_k(S^n)$, then $\{x_1, \cdots, x_k\}$ is located on a great circle S^1 and a configuration obtained by rearranging $\{x_1, \cdots, x_k\}$ is equally spaced on S^1.
(3) For an equally spaced configuration $(x_1, \cdots, x_k)$, $f_{x_1,\cdots,x_k}(x) = \sum_{i=1}^k dist(x, x_i), x \in S^n$ takes the maximum exactly at $x = \bar{x}_j$ for all j.

To show these assertions we need the following theorem.

Theorem 2.1. *Let $(x_1, \cdots, x_k)$ be an equally spaced configuration on a great circle S^1 in S^n. The function $f(x) = \sum_{i=1}^k dist(x, x_i)$ takes its maximum at $x \in S^n$ when and only when $x = \bar{x}_l (1 \leq l \leq k = 2p-1)$.*

Indeed, making use of Theorem 2.1 we may show $(1), (2)$, and (3) as follows. First consider a configuration $(x_1, \cdots, x_k)$ as in Theorem 2.1, then we have $\max_{x \in S^n} f_{x_1,\cdots,x_k} = \sum_{i=1}^k dist(x_i, \bar{x}_l) = (2p^2-2p+1)\pi/(2p-1)$. It follows that $a_k(S^n) \leq (2p^2-2p+1)\pi/(2p-1)^2$ and the assertion (1) holds. Next we show the assertion (2). Suppose $\{x_1, \cdots, x_k\}$ realizes $a_k(S^n)$, i.e., $\max_{x \in S^n} f_{x_1,\cdots,x_k} = (2p^2-2p+1)\pi/(2p-1)$. Then equality holds in the above inequalities (2.3), $\{x_1, \cdots, x_k\}$ also realizes $xt_k(S^n)$, i.e., they are lying on a great circle S^1 except for some antipodal pairs by Theorem 1.3. Note, however, that $a_{2p-1}(S^n)$ cannot be realized if $\{x_1, \cdots, x_k\}$ contains an antipodal pair. Indeed, suppose that $\{x_1, \cdots, x_{2m-1}\} (1 \leq m < p)$ is on S^1 and $\{x_{2m}, \cdots, x_{2p-1}\}$ consists of $(p-m)$ antipodal pairs. Then we have

$$\begin{aligned}
\max_{x \in S^n} f_{x_1,\cdots,x_{2p-1}}(x) &\geq \max_{x \in S^1} f_{x_1,\cdots,x_{2p-1}}(x) \\
&\geq (p-m)\pi + (2m-1)a_{2m-1}(S^1) \\
&= (p-m)\pi + \frac{2m^2-2m+1}{2m-1}\pi \\
&= \frac{(2p-1)m+1-p}{2m-1}\pi \\
&> \frac{2p^2-2p+1}{2p-1}\pi.
\end{aligned}$$

It follows that $\{x_1, \cdots, x_k\}$ is located on a great circle S^1. We may also assume that $(x_1, \cdots, x_k)$ is a configuration in S^1. Then $f_{x_1,\cdots,x_k}(x)$ attains the maximum $ka_k(S^1) = \frac{2p^2-2p+1}{2p-1}\pi$ at $x = \bar{x}_j \in S^1$ by (2.3). By restricting $f_{x_1,\cdots,x_k}$ to S^1 we see that $(x_1, \cdots, x_k)$ is equally spaced in S^1 by Theorem 1.1. Then assertion (3) also follows from Theorem 2.1.

Now we give some remarks about a proof of Theorem 2.1. It suffices to consider the case $n = 2$ to prove Theorem 2.1. By virtue of Theorem 1.1 we need only to show that $f_{x_1,\cdots,x_k}(x)$ cannot take a maximum at a point $x \in S^2 \backslash S^1$. For $k = 3$ we gave a proof of the theorem by showing that $f_{x_1,\cdots,x_k}(x)$ admits no critical points in $S^2 \backslash S^1$(see section 3 of [So]). But for $k = 2p-1 > 3$ the behavior of critical points of $f_{x_1,\cdots,x_k}(x)$ is rather complicated and it is not so clear whether the above approach works for general $k = 2p-1$. Then K. Kiyohara gave a simple and ingenious proof of theorem 2.1 which will be presented in the appendix.

3. PROOF OF THEOREM 1.5

Let X be an n-dimensional Alexandrov space with curvature ≥ 1. Recall that we have an inequality

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$$a_{2p-1}(X) \leq a_{2p-1}(S^n) = \frac{2p^2 - 2p + 1}{(2p-1)^2}\pi \tag{3.1}$$

by Theorem 1.2 and Theorem 1.4. In this section we show that X is isometric to the round sphere S^n of radius 1 when equality holds in (3.1). First we recall the notion of the spherical suspension and the spherical join([B-G-P]).

Definition 3.1. *The spherical suspension of a metric space Y is the quotient space*

$$\Sigma_1 Y = Y \times [0, \pi]/ \sim, \tag{3.2}$$

where the equivalence relation $\sim$ is given by $(x_1, a_1) \sim (x_2, a_2) \Leftrightarrow x_1 = x_2, 0 < a_1 = a_2 < \pi$ or $a_1 = a_2 = 0$ or $a_1 = a_2 = \pi$, and is equipped with the canonical metric

$$\cos dist(\hat{x}_1, \hat{x}_2) = \cos a_1 \cos a_2 + \sin a_1 \sin a_2 \cos dist(x_1, x_2), \tag{3.3}$$

where we set $\hat{x}_1 = (x_1, a_1)$, $\hat{x}_2 = (x_2, a_2)$.

Definition 3.2. *The spherical join of X and Y is defined as*

$$X * Y = X \times Y \times [0, \pi/2] / \sim, \tag{3.4}$$

where $(x_1, y_1, a_1) \sim (x_2, y_2, a_2) \Longleftrightarrow x_1 = x_2, y_1 = y_2, 0 < a_1 = a_2 < \pi/2$ or $a_1 = a_2 = 0$, $x_1 = x_2$ or $a_1 = a_2 = \pi/2$, $y_1 = y_2$, and is equipped with the canonical metric

$$\begin{aligned} &\cos dist((x_1, y_1, a_1), (x_2, y_2, a_2)) \\ &= \cos a_1 \cos a_2 \cos dist(x_1, x_2) + \sin a_1 \sin a_2 \cos dist(y_1, y_2). \end{aligned} \tag{3.5}$$

Further, we define $\Sigma_k Y = \Sigma_{k-1}(\Sigma_1 Y)$ to be the k-times repeated spherical suspension. Then for an Alexandrov space X with curvature ≥ 1 we have $X = \Sigma_k Y$ for some Alexandrov space Y with curvature ≥ 1 if and only if S^{k-1} is isometrically embedded in X([G-W]). Hence the k-times repeated spherical suspension $\Sigma_k Y$ is isometric to the spherical join $S^{k-1} * Y$.

In the previous paper, in the case of $k = 3$ we showed that X is isometric to $\Sigma_2 Z$ for some Alexandrov space Z with curvature ≥ 1. First we show that X is isometric to the spherical suspension $\Sigma_1 Y$ in the same manner as in the case of $k = 3$ for completeness.

Lemma 3.1. *Let X be an n-dimensional Alexandrov space with curvature ≥ 1. Suppose $a_{2p-1}(X) = a_{2p-1}(S^n) = \frac{2p^2-2p+1}{(2p-1)^2}\pi$. Then X is isometric to the spherical suspension $\Sigma_1 Y$, where Y is an $(n-1)$-dimensional Alexandrov space with curvature ≥ 1.*

Proof. By the maximal diameter theorem([G-P2]), it suffices to show that $diamX$ is equal to π. Let $\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_{2p-1}$ be points on S^n that realize $a_{2p-1}(S^n)$. We may assume that $(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_{2p-1})$ is an equally spaced configuration in a great circle S^1. Take a point $\tilde{p} \in S^n$ different from the antipodals of $\tilde{x}_i (i = 1, 2, \cdots, 2p-1)$. Take a regular point $p \in X$. We denote by S_p the space of directions of X at p. Then $\Sigma_1 S_p$ is isometric to S^n, and we identify $\Sigma_1 S_p$ (resp. S_p) with $S^n = \Sigma_1 S_{\tilde{p}}$ (resp. $S_{\tilde{p}} = S^{n-1}$). Let $x_i \in X$ be a point $\exp_p \bar{c}_{v_i}(dist(\tilde{p}, \tilde{x}_i)) = c_{v_i}(dist(\tilde{p}, \tilde{x}_i))$, where $\bar{c}_{v_i}$ is a minimal geodesic in S^n emanating from $\tilde{p}$ to $\tilde{x}_i$ with initial direction $v_i \in S_{\tilde{p}} = S_p = S^{n-1} (i = 1, 2, \cdots, 2p-1)$ and c_{v_i} is a quasigeodesic in X emanating from p with initial direction v_i(see section 5 of [So]). Take a point $x_0 \in X$ such that

$$\begin{aligned} a_{2p-1}(x_1, x_2, \cdots, x_{2p-1}) : &= \max_{x \in X} \frac{1}{2p-1} \sum_{i=1}^{2p-1} dist(x, x_i) \\ &= \frac{1}{2p-1} \sum_{i=1}^{2p-1} dist(x_0, x_i). \end{aligned}$$

Let $\gamma_0 : [0, dist(p, x_0)] \longrightarrow X$ be a minimal geodesic from p to x_0, and set $\tilde{x}_0 = \exp_{\tilde{p}}^{S^n}(dist(p, x_0)\dot{\gamma}_0(0))$. Then by the generalized Toponogov comparison theorem for $\triangle p x_i x_0$ and $\triangle \tilde{p} \tilde{x}_i \tilde{x}_0$([Pe], see also [So]), we have

$$dist(x_0, x_i) \le dist(\tilde{x}_0, \tilde{x}_i) \quad (i = 1, 2, \cdots, 2p-1).$$

Then we have

$$\begin{aligned} a_{2p-1}(X) \le a_{2p-1}(x_1, x_2, \cdots, x_{2p-1}) &= \frac{1}{2p-1} \sum_{i=1}^{2p-1} dist(x_0, x_i) \\ &\le \frac{1}{2p-1} \sum_{i=1}^{2p-1} dist(\tilde{x}_0, \tilde{x}_i) \le a_{2p-1}(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_{2p-1}) \\ &= a_{2p-1}(S^n) = a_{2p-1}(X). \end{aligned} \tag{3.6}$$

It follows that

$$a_{2p-1}(X) = a_{2p-1}(S^n) = \frac{1}{2p-1} \sum_{i=1}^{2p-1} dist(\tilde{x}_0, \tilde{x}_i), \tag{3.7}$$

and we obtain for any i

$$dist(x_0, x_i) = dist(\tilde{x}_0, \tilde{x}_i) \quad (i = 1, 2, \cdots, 2p-1). \tag{3.8}$$

Then from Theorem 1.1 $\tilde{x}_0$ is the antipodal point of some $\tilde{x}_i$, namely, we have $dist(\tilde{x}_0, \tilde{x}_i) = \pi$, and hence $dist(x_0, x_i) = \pi$ for some $x_i (1 \le i \le 2p-1)$. □

The following lemma is given in the previous paper([So]).

Lemma 3.2. *Suppose $X = \Sigma_1 Y$, where Y is an $(n-1)$-dimensional Alexandrov space with curvature ≥ 1 and $diam Y < \pi$ and $n \geq 2$. Let x_1,$x_2 \in X$ be the pole points of the spherical suspension $X = \Sigma_1 Y$. Then there is no pair of points whose distance is π except for x_1,x_2.*

Next we show that X is isometric to $\Sigma_2 Z$ if $\dim X \geq 2$ as in the case of $k = 3$. By Lemma 3.2 we have the following lemma.

Lemma 3.3. *Let X be an n-dimensional Alexandrov space with curvature ≥ 1 and $n \geq 2$. Suppose $a_{2p-1}(X) = a_{2p-1}(S^n) = \frac{2p^2-2p+1}{(2p-1)^2}\pi$. Then X is isometric to $\Sigma_2 Z$, where Z is an $(n-2)$-dimensional Alexandrov space with curvature ≥ 1.*

Proof. By Lemma 3.1 we may write $X = \Sigma_1 Y$. Suppose $diam Y < \pi$. In the proof of Lemma 3.1 we may take a point p as an arbitrary regular point of X, and the set of regular points is dense in X. If the base point $p \in X$ is shifted, the points $x_1, x_2, \cdots, x_{2p-1}$ that realize $a_{2p-1}(X)$ can be moved. Then there exists another pair of points $x_0, x_i (i = 1, 2, \cdots, 2p-1)$ whose distance is equal to π. This contradicts Lemma 3.2. Therefore, we have $diam Y = \pi$ and $X = \Sigma_2 Z$. □

We prepare one more lemma.

Lemma 3.4. *Let X be an n-dimensional Alexandrov space with curvature ≥ 1. Suppose $a_{2p-1}(X) = a_{2p-1}(S^n) = \frac{2p^2-2p+1}{(2p-1)^2}\pi$. If X is isometric to $S^{k-1} * Y^{n-k} (1 \leq k \leq n-1)$, where Y^{n-k} is an $(n-k)$-dimensional Alexandrov space with curvature ≥ 1, then $diam Y^{n-k} = \pi$.*

Proof. Suppose $diam Y^{n-k} < \pi$. Take any points $(x, y, t_1), (x', y', t_2) \in X (x, x' \in S^{k-1}, y, y' \in Y^{n-k}, 0 \leq t_1, t_2 \leq \pi/2)$, and set

$$l = dist((x, y, t_1), (x', y', t_2)).$$

Then we will show that $l = \pi$ holds exactly when $t_1 = t_2 = 0$ and $dist(x, x') = \pi$, namely, $(x, y, t_1), (x', y', t_2)$ are antipodal pair of S^{k-1}. Indeed, by the distance formula (3.5) we have

$$\begin{aligned} -1 &= \cos l \\ &= \cos t_1 \cos t_2 \cos dist(x, x') + \sin t_1 \sin t_2 \cos dist(y, y') \qquad (3.9) \\ &\geq \cos(\pi + t_1 - t_2) \geq -1. \end{aligned}$$

Then we have $t_1 = t_2 = 0$ because of $\cos(dist(y, y')) > -1$, and also $dist(x, x') = \pi$ holds.

Now in the proof of Lemma 3.1 we may choose a point p arbitrarily as long as p is regular. Since $rad X = a_1(X) \geq a_{2p-1}(X) = (2p^2 - 2p + 1)\pi/(2p-1)^2 > \pi/2$, X is homeomorphic to S^n by the radius sphere theorem, and the set S of non regular points is a closed set of dimension $\leq n-2$([G-P1]). Also we may choose identification between S_p and $S_{\tilde{p}} = S^{n-1}$ up to isometries of S^{n-1}. If the point $p \in X$ is shifted or $v_1, v_2, \cdots, v_{2p-1}$ are rotated around p in $S_p = S^{n-1}$, the points $x_1, x_2, \cdots, x_{2p-1}$ that realize $a_{2p-1}(X)$ can be moved outside of S^{k-1}. Then there exists another pair of points $x_0, x_i (i = 1, 2, \cdots, 2p-1)$ with $dist(x_0, x_i) = \pi$ such that they are not antipodal pair in S^{k-1}. This contradicts $diam Y^{n-k} < \pi$. It follows that $diam Y^{n-k} = \pi$ holds. □

Now we show that X is isometric to S^n.

Proof of Theorem 1.5. By Lemma 3.1 and Lemma 3.3 we have $X = S^0 * Y^{n-1} = S^1 * Y^{n-2}$. Next we assume that $X = S^{k-1} * Y^{n-k}$ holds for $k (1 \leq k \leq n-1)$. By Lemma 3.4 we have $diam Y^{n-k} = \pi$. It follows that $X = S^k * Y^{n-k-1}$. By induction on k we see that X is isometric to $S^{n-2} * Y^1$. Since $rad X = rad(\Sigma_1 Y^{n-1}) = rad Y^{n-1}$, we have $rad Y^{n-1} > \pi/2$(see [G-P1]). It follows that $rad Y^1 > \pi/2$ and Y^1 is homeomorphic to the circle S^1. By Lemma 3.4 $diam Y^1 = \pi$ and therefore Y^1 is isometric to S^1. It follows that X is isometric to S^n. This completes the proof of Theorem 1.5. □

Remark 3.1. *If $a_{2p-1}(X)$ is close to $a_{2p-1}(S^n) = (2p^2-2p+1)\pi/(2p-1)^2 > \pi/2$, then X is homeomorphic to S^n since $rad X > \pi/2$.*

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References

[B-G-P] Y. Burago-M. Gromov-G. Perelman. *Alexandrov spaces with curvature bounded below I.* Russ. Math. Surveys.**47**, 1-58(1992).

[G-M] K. Grove-S. Markvorsen. *New extremal problems for the Riemannian recognition program via Alexandrov geometry.* J. Amer. Math. **8**, 1-28(1995).

[G-P1] K. Grove-P. Petersen. *A radius sphere theorem.* Invent. Math. **112**, 577-583(1993).

[G-P2] K. Grove-P. Petersen. *On the excess of metric spaces and manifolds.* Preprint.

[G-W] K. Grove- F. Wilhelm. *Hard and soft packing theorems.* Ann. of Math.**142**, 213-237(1995).

[K] K. Kiyohara. *Appendix to "Some metric invariants of spheres and Alexandrov spaces II".* to appear in Math. J. Okayama Univ. **47**.

[P] G. Perelman. *Alexandrov spaces with curvature bounded below II.* Preprint.

[Pe] A. Petrunin. *Quasigeodesics in multidimensional Alexandrov spaces.* Diploma, University of Illinois(1995).

[P-P] G. Perelman- A. Petrunin. *Extremal subsets in Alexandrov spaces and the generalized Lieberman theorem.* St. Petersburg Math J. **5**, 215-227(1994).

[S] S. Shteingold. *Covering Radii and Paving Diameters of Alexandrov Spaces.* J. Geom. Anal. **8**, 613-627(1998).

[So] N. Sochi. *Some metric invariants of spheres and Alexandrov spaces I.* Math. J. Okayama Univ.**46**, 163-182(2004).

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